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# Symmetries and conservation laws for Hamiltonian systems with inputs and outputs: A generalization of Noether's theorem

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The definitions of symmetries and conservation laws for autonomous (i.e. without external forces) Hamiltonian systems are generalized to Hamiltonian systems with inputs and outputs. It is shown that a symmetry implies the existence of a conservation law and vice versa; thereby generalizing Noether's theorem for autonomous Hamiltonian systems.

**Keywords:** Nonlinear input-output systems, Symmetries, Hamiltonian systems, Symplectic geometry.

## Introduction

In the context of nonlinear Hamiltonian systems without external forces the study of *symmetries* is a very important and elaborated issue (cf. [1,2]). The main reason for the importance of symmetries is that they are equivalent to the existence of *conservation laws*, i.e. functions of the state which remain unaltered in time (this is usually called Noether's theorem). The content of this paper is to generalize the concept of symmetries and conservation laws to systems which are *not* isolated from the outside world, loosely speaking systems with *inputs* and *outputs*. The situation can be explained by the following simple example. Consider a particle in  $\mathbb{R}^3$  with mass  $m$  in a potential field  $V$ , and also subject to an external force  $F$ . In input-output form the system is given by

$$m\ddot{q}_i = \frac{\partial V}{\partial q_i} + F_i, \quad i = 1, 2, 3,$$

$$y_i = q_i \quad (y \text{ is the output}).$$

Suppose the equations  $m\ddot{q}_i = \partial V / \partial q_i$  are invariant under rotation around the  $e_1$ -axis (this is equivalent to  $V$  or  $L := \frac{1}{2} m \dot{q}^2 - V$  invariant). Then we know that for  $F = 0$ , the angular momentum

around the  $e_1$ -axis is preserved, i.e.

$$\frac{dI}{dt} = 0, \quad \text{with } I := \langle y \times m\dot{q}, e_1 \rangle.$$

However, for a nonzero external force we obtain

$$\frac{dI}{dt} = \langle y \times F, e_1 \rangle = \langle y \times F, e_1 \rangle.$$

Now  $P := \langle y \times F, e_1 \rangle$  is a function on the space of inputs and outputs

$$W := \{(y, F) | y \in \mathbb{R}^3, F \in \mathbb{R}^3\}.$$

With the natural symplectic form

$$\begin{bmatrix} 0 & -I_3 \\ I_3 & 0 \end{bmatrix}$$

on  $W$ , the function  $P$  induces a Hamiltonian vectorfield  $X_P$  on  $W$ . In coordinates:

$$P = y_2 F_3 - F_2 y_3,$$

and  $X_P$  is given by

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \\ \frac{d}{dt} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}. \end{aligned}$$

$X_P$  can be interpreted as an *external symmetry*, which expresses the fact that the output corresponding to an external force which is rotated around the  $e_1$ -axis is obtained by rotating the output in the same way.

**Notation.** We assume that the reader is familiar with the basic notions of differential geometry (see [10,1]). For *symplectic geometry* we refer to [1,2,7].  $L_X$  will denote the Lie derivative with respect to a vectorfield  $X$ . Given a function  $H$  on a symplectic manifold  $(M, \omega)$ , we will denote the Hamiltonian

vectorfield  $X$  defined by  $\omega(\dot{X}, \cdot) = dH$  by  $X_H$ . Suppose  $N$  is a Lagrangian submanifold of  $(M, \omega)$ . Given canonical coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n)$  for  $M$  ( $2n$ -dimensional; canonical means  $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ ), we can find a function  $S(q_1, \dots, q_n)$  such that  $N$  is locally described by

$$p_i = \frac{\partial S}{\partial q_i}, \quad i = 1, \dots, n$$

(this is true if  $N$  can be parametrized by  $q_1, \dots, q_n$ ). We will call  $S$  a *generating function* of  $N$ . Given a vectorfield  $X$  on  $M$  we can define the prolongation  $\tilde{X}$ , a vectorfield on  $TM$ , as follows (see [5]).  $\tilde{X}$  generates a one-parameter group  $X_t: M \rightarrow M$  ( $t$  small), given by the integral curve of  $X$ . Then  $(X_t)_*: TM \rightarrow TM$  is again a one-parameter group corresponding to a vectorfield on  $TM$ , which we will denote by  $\tilde{X}$ .

### 1. Symmetries for nonlinear systems

We will consider nonlinear systems which in local coordinates have the form

$$\begin{aligned} \dot{x} &= g(x, v), \\ w &= h(x, v), \end{aligned} \quad (1.1)$$

with  $x$  the state-space variables,  $w$  the *external variables* (think of outputs and inputs) and  $v$  initially seen as dummy variables which generate the possible trajectories of the system. The more usual input-output framework  $\dot{x} = g(x, u)$ ,  $y = h(x, u)$  is a slight specialization of (1.1) see [8, 11]. Coordinate-free we can describe (1.1) by the commutative diagram (due to Willms [11], and in a slightly more restricted set-up, to Brockett [3, 12])

$$\begin{array}{ccc} B & \xrightarrow{f} & TM \times W \\ \pi \searrow & & \swarrow \pi_M \\ & M & \end{array} \quad (1.2)$$

where  $M$  is the state space,  $W$  the external signal space and  $B$  a fiber-bundle above  $M$  with projection  $\pi$ , and  $\pi_M$  the natural projection of  $TM$  on  $M$ . When we take coordinates  $x$  for  $M$ ,  $w$  for  $W$  and  $v$  for the fibers of  $B$  and write  $f$  as  $(g, h)$ , we arrive again at (1.1).

The following notions are adapted from [11]. We will denote system (1.2) by  $\Sigma(M, W, B, f)$  or shortly  $\Sigma$ . The *internal behavior*  $\Sigma_i$  of  $\Sigma(M, W, B, f)$  (the collection of state/external signal trajectories)

is given by

$$\Sigma_i := \{(x(\cdot), w(\cdot)): \mathbb{R} \rightarrow M \times W \mid \dot{x}(\cdot) \text{ abs. cont. and } (x(t), w(t)) \in f(\pi^{-1}(x(t))) \text{ a.e.}\}.$$

The *external behavior*  $\Sigma_e$  is just the projection of  $\Sigma_i$  on  $W^{\mathbb{R}}$ , i.e.

$$\Sigma_e := \{w(\cdot): \mathbb{R} \rightarrow M \times W \mid \exists x(\cdot): \mathbb{R} \rightarrow M \text{ such that } (x(\cdot), w(\cdot)) \in \Sigma_i\}.$$

If for a given  $\Sigma_e$  we construct a  $\Sigma(M, W, B, f)$  with  $\Sigma_e$  as its external behavior we call  $\Sigma(M, W, B, f)$  a *state space realization* of  $\Sigma_e$ . Intuitively we will call  $\Sigma$  a *minimal realization* of  $\Sigma_e$  if there exists no 'smaller' realization  $\Sigma'$  of  $\Sigma_e$ . More precise, let  $\Sigma(M, W, B, f)$  and  $\Sigma'(M', W', B', f')$  be two systems. Then we say  $\Sigma' \leq \Sigma$  if there exist surjective submersions  $\varphi: M \rightarrow M'$ ,  $\phi: B \rightarrow B'$  such that the diagram

$$\begin{array}{ccccc} B & & \xrightarrow{\phi} & & B' \\ \pi \downarrow & \nearrow f & & \nwarrow f' & \downarrow \pi' \\ & W & \xleftarrow{\text{id}} & W & \\ & \times & & \times & \\ & TM & \xleftarrow{\varphi_*} & TM' & \\ \pi_M \swarrow & & & & \searrow \pi'_M \\ M & & \xrightarrow{\varphi} & & M' \end{array} \quad (1.3)$$

commutes,  $\Sigma$  and  $\Sigma'$  are called *equivalent* (denoted  $\Sigma \sim \Sigma'$ ) if  $\varphi$  and  $\phi$  are diffeomorphisms. We call  $\Sigma$  *minimal* if  $\Sigma' \leq \Sigma \Rightarrow \Sigma' \sim \Sigma$  (see [7]). These notions in our framework are adaptations of the work on differential geometric systems by Brockett, Sussman, Hermann and Krener. See [12] for a systematic exposition of this.

Now we will proceed to the definitions of a *symmetry*.

**Definition 1.1** (see [7]). Let there be given a nonlinear system  $\Sigma(M, W, B, f)$  with external behavior  $\Sigma_e$ . An *external symmetry* is a diffeomorphism  $\psi: W \rightarrow W$  such that  $\psi$  leaves  $\Sigma_e$  invariant, i.e.  $\psi(\Sigma_e) = \Sigma_e$ , where  $\psi: W^{\mathbb{R}} \rightarrow W^{\mathbb{R}}$  is in the obvious way induced by  $\psi$ .

The following is immediate:

**Proposition 1.2.** Let  $\Sigma(M, W, B, f)$  be a system. Denote  $f: B \rightarrow TM \times W$  by the components  $(g, h)$ . Let

$\psi: W \rightarrow W$  be an external symmetry. Then  $\Sigma(M, W, B, f)$  with  $\tilde{f}$  defined by  $\tilde{f} = (g, \psi \circ h)$  realizes the same external behavior as  $\Sigma$ , i.e.  $\Sigma_{\tilde{f}} = \Sigma_{\tilde{g}}$ . Moreover  $\Sigma$  is minimal iff  $\Sigma$  is minimal.

An internal (external and state-space) symmetry is defined as follows:

**Definition 1.3.** (see [7]). Let  $\Sigma(M, W, B, f)$  be a system. An internal symmetry or shortly, symmetry, is a 3-tuple  $(\phi, \varphi, \psi)$ , with  $\phi: B \rightarrow B$ ,  $\varphi: M \rightarrow M$  and  $\psi: W \rightarrow W$  diffeomorphisms, such that the diagram

$$\begin{array}{ccccc}
 B & \xrightarrow{\phi} & B & & \\
 \downarrow f & & \downarrow f & & \\
 W & \xrightarrow{\psi} & W & & \\
 \downarrow \times & & \downarrow \times & & \\
 TM & \xrightarrow{\varphi_*} & TM & & \\
 \downarrow \pi_M & & \downarrow \pi_M & & \\
 M & \xrightarrow{\varphi} & M & & \\
 \text{commutes.} & & & & 
 \end{array} \quad (1.4)$$

The following proposition is immediate:

**Proposition 1.4.** Let  $(\phi, \varphi, \psi)$  be an internal symmetry for  $\Sigma$ , then  $\psi$  is an external symmetry for  $\Sigma_{\phi}$ .

**Remark.** We should also like to have the inverse statement that given an external symmetry  $\psi$  we can construct  $\phi$  and  $\varphi$  such that  $(\phi, \varphi, \psi)$  is an internal symmetry. However, only for externally induced state-space realizations  $\Sigma(M, W, B, f)$  (see [11]) this can be easily proven. The general case gives problems which are related with the essential non-uniqueness of minimal nonlinear state-space realizations (see [11]), and we will leave it for the moment.

In the study of symmetry it is of interest to study symmetry groups. For smooth one-parameter groups we may then as well consider the infinitesimal versions to which Definition 1.1 and 1.3 are readily generalized. We shall only consider the infinitesimal version of Definition 1.3:

**Definition 1.5.** Let  $\Sigma(M, W, B, f)$  be a system. An infinitesimal symmetry is a 3-tuple  $(R, S, T)$ , with

$R, S$  and  $T$  vectorfields on  $B, M$ , resp.  $W$ , such that the diagram

$$\begin{array}{ccccc}
 B & \xrightarrow{R_t} & B & & \\
 \downarrow f & & \downarrow f & & \\
 W & \xrightarrow{T_t} & W & & \\
 \downarrow \times & & \downarrow \times & & \\
 TM & \xrightarrow{S_{*t}} & TM & & \\
 \downarrow \pi_M & & \downarrow \pi_M & & \\
 M & \xrightarrow{S_t} & M & & 
 \end{array} \quad (1.5)$$

commutes. Here  $R_t, S_t$  and  $T_t$  ( $t \in \mathbb{R}$  and small) denote the one-parameter groups generated by  $R, S$ , resp.  $T$ .

It turns out that the commutativity of this diagram can be expressed in a very concise way.

**Proposition 1.6.** Let  $\Sigma(M, W, B, f)$  be a system and denote  $\tilde{f}$  by  $\tilde{f} = (g, h)$ . Then the commutativity of (1.5) is equivalent to

- (i)  $g_* R = \tilde{S}$ ,
- (ii)  $h_* R = T$ .

**Proof.** From (1.5) it follows that

$$S_{*t} \circ g = g \circ R_t,$$

for every  $t$  small. Differentiating this expression with respect to  $t$  gives

$$g_* R = \tilde{S}$$

(see the notation for the definition of  $\tilde{S}$ ). Analogously it follows from (1.5) that  $T_t \circ h = h \circ R_t$ , for every  $t$  small. Differentiating gives  $h_* R = T$ .  $\square$

**Remark.** For locally minimal systems (somewhat stronger than minimal, see [8]) we can state the following:

Let  $(R, S, T)$  and  $(\tilde{R}, \tilde{S}, \tilde{T})$  be infinitesimal symmetries. Then  $R = \tilde{R}$ ,  $S = \tilde{S}$  (this can easily be proved following the methods in [8]; hence  $R$  and  $S$  are uniquely determined by  $T$ ).

## 2. Symmetries for Hamiltonian systems

We will briefly repeat the definition of a nonlinear system as given in [7] (see also [3]).

**Definition 2.1.**  $\Sigma(M, W, B, f)$  is a full Hamiltonian system if

- (i)  $M$  is a manifold with symplectic form  $\omega$ ,
- (ii)  $W$  is a manifold with symplectic form  $\omega^*$
- (iii)  $f(B)$  is a Lagrangian submanifold of the manifold  $TM \times W$  with symplectic form  $\pi_1^* \omega + \pi_2^* \omega^*$  ( $\pi_1$ , resp.  $\pi_2$  projection of  $TM \times W$  on  $TM$ , resp.  $W$ ), where  $\tilde{\omega}$  is the canonically induced symplectic form on  $TM$ .

In local coordinates this gives:

**Proposition 2.2** [7]. Let  $\Sigma(M, W, B, f)$  be a full Hamiltonian system, then there exist coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n)$  for  $M$  and coordinates  $(y_1, \dots, y_m, u_1, \dots, u_m)$  for  $W$  and a function  $H(q_1, \dots, q_n, p_1, \dots, p_n, u_1, \dots, u_m)$  such that the system is described by

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q, p, u), & i &= 1, \dots, n, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q, p, u), \\ y_j &= c_j \frac{\partial H}{\partial u_j}(q, p, u), & j &= 1, \dots, m, \quad c_j = \pm 1 \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} \omega &= \sum_{i=1}^n dq_i \wedge dp_i, \\ \omega^* &= \sum_{j=1}^m c_j dy_j \wedge du_j. \end{aligned}$$

The symplectic structure as in Definition 2.1 is mirrored in the following definition of a symmetry for a Hamiltonian system:

**Definition 2.3** Let  $\Sigma(M, W, B, f)$  be a Hamiltonian system. An internal symmetry  $(\phi, \varphi, \psi)$  is a Hamiltonian symmetry if  $\phi$  and  $\psi$  are symplectomorphisms, i.e.

- (i)  $\phi^* \omega = \omega$ ,
  - (ii)  $\psi^* \omega^* = \omega^*$
- ( $\omega$ , resp.  $\omega^*$  is the symplectic form on  $M$ , resp.  $W$ ).

Actually, we don't have to assume a priori for a minimal system that  $\phi$  is a symplectomorphism, as is shown by the following proposition:

**Proposition 2.4.** Let  $\Sigma(M, W, B, f)$  be a full Hamiltonian and minimal system. Let  $(\phi, \varphi, \psi)$  be an

internal symmetry. Furthermore, let  $\psi$  be a symplectomorphism. Then  $\phi$  is necessarily also a symplectomorphism.

**Proof.** When we write  $f = (g, h)$ , then  $f(B)$ , Lagrangian with respect to  $\pi_1^* \omega + \pi_2^* \omega^*$ , is equivalent to

$$g^* \tilde{\omega} = h^* \omega^*. \quad (2.2)$$

Because  $(\phi, \varphi, \psi)$  is a symmetry,  $f(B)$  is mapped by  $\psi$  and  $\varphi_*$  onto  $f(B)$ . Therefore  $\Sigma(M, W, B, f)$  with  $\tilde{f} := (\varphi_* \circ g, \psi_* \circ h)$  is again a Hamiltonian system. Hence

$$g^*((\varphi_*)^* \tilde{\omega}) = h^*(\varphi^* \omega^*).$$

Therefore

$$g^* \tilde{\omega} = h^* \omega^* = h^* \psi^* \omega^* = g^*((\varphi_*)^* \tilde{\omega}) \quad (2.3)$$

where we use  $\psi^* \omega^* = \omega^*$ .

When we denote  $\tilde{\omega} := \omega - \varphi^* \omega$ , then (2.3) gives  $g^* \tilde{\omega} = 0$ .

Using the fact that the system consists of Hamiltonian vectorfields, and therefore the Lie brackets of those vectorfields are again Hamiltonian, we can prove, in a manner analogous to [8], that because the system is minimal and therefore strongly accessible, this implies that  $\tilde{\omega} = 0$ , or  $\omega = \varphi^* \omega$ .  $\square$

From now on we want to concentrate on infinitesimal Hamiltonian symmetries.

**Definition 2.5.** An infinitesimal symmetry  $(R, S, T)$  of a Hamiltonian system  $\Sigma(M, W, B, f)$  is called Hamiltonian if  $S$  and  $T$  are (locally) Hamiltonian vectorfields, i.e.  $L_S \omega = 0$  and  $L_T \omega^* = 0$ .

**Remark.** From Proposition 2.4 it follows that when  $\Sigma(M, W, B, f)$  is full Hamiltonian and minimal, then  $L_T \omega^* = 0$  implies  $L_S \omega = 0$ .

For Hamiltonian infinitesimal symmetries we can deduce the following (main) theorem:

**Theorem 2.6.** Let  $\Sigma(M, W, B, f = (g, h))$  be Hamiltonian. Let  $(R, S, T)$  be an infinitesimal Hamiltonian symmetry. Let  $(x, u)$  be coordinates for  $B$ , with  $x$  coordinates for  $M$ , such that  $g(x, u)$  is for fixed  $u$  a Hamiltonian vector on  $M$  (such coordinates exist by Proposition 2.2).

Then we can (locally) construct functions  $F: M \rightarrow \mathbb{R}$  and  $F^*: W \rightarrow \mathbb{R}$  such that  $S = X_F$  and  $T = X_{F^*}$  and

$$g(F) = i^* \circ h \quad (2.4)$$

where  $g: E \rightarrow TM$  is interpreted as the collection of Hamiltonian vectorfields given for every constant  $u$  by  $g(\cdot, u)$ .

Proof. We can deduce the following equalities from (1.6) and (2.2):

$$\begin{aligned} i^* \omega &= h^* \omega^* \Rightarrow g^* \omega(R_\cdot) = h^* \omega^*(R_\cdot) \\ &\Leftrightarrow \dot{\omega}(g_\cdot R_\cdot, g_\cdot) = \omega^*(h_\cdot R_\cdot, h_\cdot) \\ &\Leftrightarrow \dot{\omega}(S, g_\cdot) = \omega^*(T, h_\cdot) \quad (\text{by (1.6)}) \\ &\Leftrightarrow d\dot{H}(g_\cdot) = dF^*(h_\cdot), \end{aligned} \quad (2.5)$$

where  $F$  is a local Hamilton function of  $S$  (i.e.  $S = X_F$ ),  $F^*$  a local Hamilton function of  $T$  (i.e.  $T = X_{F^*}$ ) and where we use the fact that  $\dot{\omega}(S_\cdot) = \dot{\omega}^*(h_\cdot)$  when  $\omega(S_\cdot) = dF$  (see [5]).

Following Proposition 2.2 we can find coordinates  $(q, p, u)$  for  $B$  and locally a function  $H(q, p, u)$  such that

$$g(q, p, u) = \begin{pmatrix} \frac{\partial H}{\partial p}(q, p, u) \\ -\frac{\partial H}{\partial q}(q, p, u) \end{pmatrix}.$$

Furthermore,

$$F = \sum_i \left[ \frac{\partial F}{\partial g_i} q_i + \frac{\partial F}{\partial p_i} p_i \right]$$

and hence

$$\begin{aligned} F \circ g &= \sum_i \left[ -\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right] \\ &= \{F, H\} = g(F) \end{aligned}$$

where  $\{ \cdot, \cdot \}$  is the Poissonbracket on  $M$ . Therefore (2.5) is equivalent to

$$d(g(F)) = d(F^* \circ h).$$

When we use the freedom in the choice of  $F^*$  (uniquely defined by  $T$  up to a constant) we obtain

$$g(F) = F^* \circ h. \quad \square$$

Expression (2.4) can be regarded as a generalization of the usual conservation laws for autonomous (i.e. without inputs) Hamiltonian systems. In that case the usual framework is as follows.

Let  $X_H$  be a Hamiltonian vectorfield. A vectorfield  $S$  is a symmetry for  $X_H$  if

$$(i) L_S \omega = 0,$$

$$(ii) S(H) = 0$$

(see e.g. [1,2]). Now because  $L_S \omega = 0$ , there exists locally a function  $F$  such that  $S = X_F$ . Then (ii) implies

$$X_H(F) = 0. \quad (2.6)$$

so  $F$  is a conservation law (see [1,2]).

In our case symmetries satisfy (1.6) and as an analogon of (2.6) we obtain (2.4) which expresses that the change of  $F$  along the integral curves of the systems is a function of the external variables, which gives what one would call a 'conservation law' for systems with inputs and outputs.

### 3. Symmetries for affine Hamiltonian systems

A widely studied special class of systems is formed by the affine systems—usually called input linear systems. Let  $\Sigma(M, W, B, f = (g, h))$  be a full Hamiltonian and minimal system. Suppose that  $B$  is a vectorbundle. Suppose further that  $W = T^*Y$  and  $h: B \rightarrow W$  is a vectorbundle morphism, then in [9] it is proven that the system is locally given by

$$\begin{aligned} \dot{x} &= A(x) + \sum_{i=1}^m u_i B_i(x), \\ Y_i &= C_i(x), \quad i = 1, \dots, m, \end{aligned} \quad (3.1)$$

where  $A$  is locally Hamiltonian ( $L_A \omega = 0$ ) and  $B_i$  are Hamiltonian vectorfields with  $\omega(B_i \cdot) = dC_i$ , and  $(y_1, \dots, y_m, u_1, \dots, u_m)$  are canonical coordinates for  $T^*Y$ . We will call (3.1) an *affine Hamiltonian system*. Let there now be given a Hamiltonian symmetry  $(R, S, T)$  for (3.1). Then formula (2.4) specializes to

$$\left( A + \sum_{i=1}^m u_i B_i \right) (F) = F^*(u, C(\cdot))$$

with  $C := (C_1, \dots, C_m)$ .

Because  $L_S \omega = 0$  there exists locally an  $H: M \rightarrow \mathbb{R}$  such that  $A = X_H$ . Because  $S = X_C$ , we obtain

$$\{H, F\} + \sum_{i=1}^m u_i \{C_i, F\} = F^*(u, C(\cdot)). \quad (3.2)$$

Because the left-hand side of (3.2) is clearly affine in  $u$ , also  $F^*$  should be affine in  $u$ . Therefore there exist functions  $V$  and  $K_i$  on  $Y$ ,  $i=1, \dots, m$  such that

$$F^*(u, C(\cdot)) = \sum_{i=1}^m u_i K_i(C(\cdot)) + V(C(\cdot))$$

and (3.2) reduces to

$$\{H, F\} = V \circ C \quad \text{and} \quad \{C_i, F\} = K_i \circ C. \quad (3.3)$$

Actually this form of  $F^*$  is very natural as is shown by the following proposition, which we state without proof.

**Proposition 3.1.** *Let  $T$  be a Hamiltonian vectorfield on  $T^*Y$  (with the natural symplectic form), which preserves the fibers of  $T^*Y$ . Then  $T$  has a Hamilton function  $F^*$  of the form*

$$F^*(u, y) = \sum_{i=1}^m u_i K_i(y) + V(y).$$

Moreover, it can be seen that when  $T$  is also tangent to  $Y$  we can take  $V=0$  or otherwise

$$\{H, F\} = 0, \quad \{C_i, F\} = K_i \circ C. \quad (3.4)$$

These equations express that  $F$  is a real conservation law: for zero external force,  $F$  is conserved along the trajectories of the system, while for nonzero external force  $(u_1, \dots, u_m)$  the change of  $F$  equals

$$\sum_i u_i \{C_i, F\} = \sum_i u_i K_i(y) \quad \text{with } y = C(x).$$

We now want to show that by applying 'Hamiltonian feedback' to the system (3.1) we can always change (3.3) into (3.4).

First we define Hamiltonian feedback (for affine systems).

**Definition 3.2.** Given an affine Hamiltonian and minimal system (3.1). A feedback  $u \rightarrow v := \alpha(x, u)$  is called *Hamiltonian* if the system after feedback is again an affine Hamiltonian system

$$\begin{aligned} \dot{x} &= \tilde{A}(x) + \sum_{i=1}^m v_i \tilde{B}_i(x), \\ v_i &= C_i(x), \quad i=1, \dots, m. \end{aligned} \quad (3.5)$$

**Theorem 3.3.** *A Hamiltonian feedback is necessarily output feedback. Furthermore,  $\tilde{A}$  and  $\tilde{B}_i$  (as in (3.5))*

*are given by*

- (i)  $\tilde{B}_i = B_i$ ,  $i=1, \dots, m$ ,  
 (ii)  $\tilde{A} = X_H$  with  $\tilde{H} = H + P \circ C$ ,  
 where  $H$  satisfies  $A = X_H$  and  $P$  is a function on  $Y$ .

**Remark.** Hence Hamiltonian feedback adds a 'potential' function, which is only a function of the output (see also [4]).

**Proof.** Because the system after feedback must be affine,  $\alpha(x, u)$  has the form  $\alpha(x, u) = v(x) - K(x)u$  with  $v$  a vector and  $K(x)$  a matrix (see [6]). Because  $\omega(\tilde{B}_i, \cdot) = dC_i$ , it follows that  $\tilde{B}_i = B_i$ ,  $i=1, \dots, m$ .

In order to prove that  $\alpha$  is output feedback we make the following observation. The choice of natural canonical coordinates  $\{y_1, \dots, y_m, u_1, \dots, u_m\}$  on  $T^*Y$  induces a coordinatization of the vectorbundle  $B$  in the following way. Because  $h$  is a vectorbundle morphism we can find coordinates  $(x_1, \dots, x_n, u_1, \dots, u_m)$  for  $B$ , with  $(x_1, \dots, x_n)$  arbitrary coordinates for  $M$  and  $\{u_1, \dots, u_m\}$  linear coordinates for the fibers of  $B$ , such that  $h: B \rightarrow T^*Y$  in these coordinates for  $B$  and  $\{y_1, \dots, y_m, u_1, \dots, u_m\}$  for  $T^*Y$  is given by  $h(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{u}_1, \dots, \tilde{u}_m) = (C_1(\tilde{x}), \dots, C_m(\tilde{x}), \tilde{u}_1, \dots, \tilde{u}_m)$ , with  $(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{u}_1, \dots, \tilde{u}_m) \in B$ .

By identifying  $u_i'$  and  $u_i$ , this gives sections of the bundle  $B$  given by  $(u_1, \dots, u_m) = \text{constant}$ . Feedback amounts to changing these sections into the new sections given by  $\alpha(x, u) = v = \text{constant}$  (for a systematic treatment of this topic see [12], followed up in [6]).

However, in the Hamiltonian case these new sections of  $B$  have to satisfy the condition that the images of these sections under  $h$  in  $W$  are Lagrangian submanifolds of  $(W, \omega^*)$  and therefore have dimension  $m$  ( $\dim W = 2m$ ). This implies that  $\alpha(x, u)$  can only depend on  $C(x)$  and  $u$ , and therefore there exists an output feedback  $v = \tilde{\alpha}(y, u)$  such that  $\alpha(x, u) = \tilde{\alpha}(C(x), u)$ . Because  $\Sigma(M, W, B, f = (g, h))$  is an affine Hamiltonian system,  $f(B)$  is a Lagrangian submanifold of  $(TM \times W, \pi_1^* \omega - \pi_2^* \omega^*)$ , which has a generating function  $H(x) + u^*y$ . The output feedback  $v = \tilde{\alpha}(y, u)$  is such that  $(y, v)$  are again canonical coordinates (i.e.  $\omega^* = \sum_{i=1}^m dy_i \wedge dv_i$ ), or equivalently the sections in  $T^*Y$  defined by  $(v_1, \dots, v_m) = \text{constant}$  are Lagrangian submanifolds of  $(T^*Y, \omega^*)$ . Therefore the section  $(v_1, \dots, v_m) = 0$  has a generating function  $P(y)$ . Hence in the new input coor-

dinates  $(v_1, \dots, v_m)$ ,  $f(B)$  has the generating function  $H(x) + P(y) + \omega^*y$ . Therefore  $\tilde{A} = X_{\tilde{H}}$  with  $\tilde{H} = H + P \circ C$ .  $\square$

Now we are able to state the following:

**Proposition 3A.** Let  $(R, S, T)$  be a Hamiltonian symmetry for (3.1), such that

$$\begin{aligned} \{H, F\} &= V \circ C, \\ \{C_i, F\} &= K_i \circ C, \quad i = 1, \dots, m, \end{aligned} \quad (3.3)$$

where  $S = X_F$ ,  $T = X_{F^*}$ , with

$$F^*(u, y) = \sum_{i=1}^m u_i K_i(y) + V(y).$$

Then we can construct a Hamiltonian feedback, given as above by  $\tilde{H} = H + P \circ C$ , such that  $\{\tilde{H}, F\} = 0$ .

**Proof.**  $T$  is a vectorfield on  $T^*Y$  which leaves the fibers invariant. Therefore  $\pi_* T$  (with  $\pi: T^*Y \rightarrow Y$  natural projection) is a vectorfield on  $Y$ . Choose coordinates  $(y_1, \dots, y_m)$  for  $Y$  such that  $\pi_* T = \partial/\partial y_i$ . Denote the natural coordinates for the fibers of  $T^*Y$  by  $(u_1, \dots, u_m)$ . Because  $T$  is Hamiltonian it is given in these coordinates by:

$$T = \frac{\partial}{\partial y_1} + \sum_{i=1}^m \frac{\partial W}{\partial y_i} \frac{\partial}{\partial u_i}, \quad \text{with } W: Y \rightarrow \mathbb{R}.$$

Now define

$$Z(y_1, \dots, y_m) := \int_0^1 W(t, y_2, \dots, y_m) dt$$

and

$$W_i = \frac{\partial Z}{\partial y_i}, \quad i = 2, \dots, m.$$

It is clear that  $\partial Z/\partial y_1 = W$ . If we denote  $W_1 := W$ , then it is immediate that

$$\frac{\partial W_i}{\partial y_j} = \frac{\partial W_j}{\partial y_i}, \quad i, j = 1, \dots, m.$$

Define the Hamiltonian vectorfields  $T_i$ ,  $i = 2, \dots, m$  by

$$T_i = \frac{\partial}{\partial y_i} + \sum_{j=1}^m \frac{\partial W_j}{\partial y_j} \frac{\partial}{\partial u_j}.$$

When we denote  $T_1 := T$ , then it is easy to calcu-

late that  $\{T_i, T_j\} = 0$ ,  $i, j = 1, \dots, m$ . Therefore we can find new input coordinates  $(v_1, \dots, v_m)$  (because  $T_i$  is Hamiltonian) such that  $T_i = X_{v_i}$ ,  $i = 1, \dots, m$ , and also

$$\omega^* = \sum_{i=1}^m dy_i \wedge dv_i.$$

These new coordinates  $(v_1, \dots, v_m)$  define, as we have seen in the proof of Theorem 3.3, a Hamiltonian feedback  $\tilde{H} = H + P \circ C$ .

Moreover, because  $T$  is tangent to the sections  $\{v_1, \dots, v_m\} = \text{constant}$ , we must have  $\{\tilde{H}, F\} = 0$  (see Proposition 3.1).  $\square$

We sum up the former results as follows:

**Theorem 3.5.** Let  $(R, S, T)$  be a Hamiltonian symmetry for an affine Hamiltonian system (3.1). Then, possibly after Hamiltonian feedback,  $(R, S, T)$  induces a real conservation law  $F$ , i.e.,

$$\{H, F\} = 0, \quad \{C_i, F\} = K_i \circ C,$$

where  $S = X_F$ ,  $T = X_{F^*}$ , with

$$F^*(y, u) = \sum_{i=1}^m u_i K_i(y).$$

(with  $(y, u)$  canonical coordinates for  $T^*Y$ , i.e.  $\omega^* = \sum_{i=1}^m dy_i \wedge du_i$ ).

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